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Postbuckling of shear deformable laminated plates resting on a tensionless elastic foundation subjected to mechanical or thermal loading

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Abstract

Postbuckling responses of shear deformable laminated plates supported by a tensionless elastic foundation and subjected to in-plane compressive edge loads or a uniform temperature rise are investigated. The formulations are based on the higher order shear deformation plate theory with a von Kármán-type of kinematic non-linearity and include the plate-foundation interaction, for which the foundation reacts in compression only. The thermal effects are also included and the material properties are assumed to be independent of temperature. The initial geometric imperfection of the plate is taken into account. The analysis uses a two step perturbation technique to determine the postbuckling response of the plate. An iterative scheme is developed to obtain numerical results without using any prior assumption for the shape of the contact region. The numerical illustrations concern the postbuckling behavior of antisymmetric angle-ply and symmetric cross-ply laminated plates resting on tensionless elastic foundations of the Pasternak-type, from which results for conventional elastic foundations are obtained as comparators. The results reveal that the unilateral constraint has a significant effect on the postbuckling response of the plate subjected to either mechanical or thermal loading when the foundation stiffness is sufficiently large.

Keywords: Contact buckling; Postbuckling; Laminated plate; Higher order shear deformation plate theory; Tensionless elastic foundation

1. Introduction

Contact between flexible plate elements and hard substrates is an issue of concern in a variety of many technological applications, particularly in analysis and design of delaminated multilayered plates that contain interlaminar defects. The local buckling of such plates is one special class of unilateral buckling

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problems. This contact buckling problem may be modeled as a thin plate resting on a tensionless elastic foundation. The solution method required to determine the response of such plates on tensionless elastic foundations is quite complicated because the contact region is not known at the outset. However, the capability to predict the postbuckling response of composite laminated plates with unilateral constraints subjected to in-plane edge compressive loads or thermal loading is of prime interest to structural analysis.

Many studies based on the classical plate theory for the contact buckling of thin plates subjected to uniaxial compression are available in the literature. Among those, Seide (1958) studied contact effects in buckled plates of infinitely long under simply supported boundary conditions with longitudinal edges immovable. This work was then extended to the case of orthotropic thin plates under simply supported and clamped-free boundary conditions by Shahwan and Waas (1998). The buckling strength of finite size plates with unilateral constraints was considered by Bezine et al. (1985), Wright (1993), Shahwan and Waas (1994) and Smith et al. (1999a,b) using finite element method (FEM) and Rayleigh–Ritz approaches. All the aforementioned studies focused on the cases of linear buckling problem and they concluded that the lateral constraint increases the buckling load. In contrast, there have been fewer investigations on the postbuckling analysis of unilaterally constrained plates. Ohtake et al. (1980) studied the postbuckling behavior of a simply supported square thin plate with unilateral constraints using a finite element scheme coupled with a penalty method. Chai (2001) studied the postbuckling behavior, including secondary buckling and snapping of a clamped thin plate unilaterally constrained by a rigid foundation. de Holanda and Gonçalves (2003) presented a postbuckling analysis of a simply supported thin plate resting on a tensionless elastic foundation. In their analysis non-linear finite element equations based on Marguerre's shallow shell theory modified by Mindlin hypothesis were formulated. In these studies only isotropic plates were considered and most of them have assumed perfectly flat initial configurations. However, to the best knowledge of the authors', studies on contact postbuckling of composite laminated plates subjected to thermal loading have not been reported in the literature.

The present paper extends the previous works (Shen, 1999, 2000a) to the case of shear deformable laminated plates supported by a tensionless elastic foundation. The temperature field considered is assumed to be a uniform distribution over the plate surface and through the plate thickness. The material properties are assumed to be independent of the temperature. The formulations are based on the higher order shear deformation plate theory with a von Kármán-type of kinematic non-linearity and include the plate-foundation interaction, for which the foundation reacts in compression only. The analysis uses a two step perturbation technique to determine the postbuckling response of the plate. An iterative scheme is developed to obtain numerical results without using any prior assumption for the shape of the contact region. The initial geometric imperfection of the plate is taken into account but, for simplicity, its form is assumed to be as the buckling mode of the plate.

2. Analytical formulations

Consider a rectangular plate of length a, width b and thickness t which consists of N plies, simply supported at four edges, and resting on an elastic foundation. The origin of the coordinate system is located at the corner of the plate in the middle plane. The plate is assumed to be geometrically imperfect, and is subjected to a compressive edge load P_x in the X-direction or exposed to a uniform temperature field. Let $\overline{U}$, $\overline{V}$ and $\overline{W}$ be the plate displacements parallel to a right-hand set of axes (X, Y, Z), where X is longitudinal and Z is perpendicular to the plate. $\overline{\Psi}_x$ and $\overline{\Psi}_y$ are the mid-plane rotations of the normals about the Y and X axes, respectively. The plate is attached to an elastic foundation of the Pasternak-type. The reaction of the foundation is assumed to be $p = \overline{K}_1\overline{W} - \overline{K}_2\nabla^2\overline{W}$, where p is the force per unit area, $\overline{K}_1$ is the Winkler foundation stiffness and $\overline{K}_2$ is a constant showing the effect of the shear interactions of the vertical elements, and ∇^2 is the Laplace operator in X and Y. This reaction, however, is only compressive and occurs only where $\overline{W}$ is positive. Denoting the initial geometric imperfection by $\overline{W}^*(X,Y)$, let $\overline{W}(X,Y)$ be the additional deflection and $\overline{F}(X,Y)$ be the stress function for the stress resultants defined by $\overline{N}_x = \overline{F}_{,yy}$, $\overline{N}_y = \overline{F}_{,xx}$ and $\overline{N}_{xy} = -\overline{F}_{,xy}$, where a comma denotes partial differentiation with respect to the corresponding coordinates.

Reddy (1984a) developed a simple higher order shear deformation plate theory, in which the transverse shear strains are assumed to be parabolically distributed across the plate thickness and which contains the same dependent unknowns as in the first order shear deformation theory, and no shear correction factors are required. From Reddy's higher order shear deformation plate theory (see Reddy, 1984b), including the plate-foundation interaction and thermal effects, the non-linear differential equations of such plates in the von Kármán sense are

$$\begin{aligned}&\widetilde{L}_{11}(\overline{W}) - \widetilde{L}_{12}(\overline{\Psi}_x) - \widetilde{L}_{13}(\overline{\Psi}_y) + \widetilde{L}_{14}(\overline{F}) - \widetilde{L}_{15}(\overline{N}^{\mathrm{T}}) - \widetilde{L}_{16}(\overline{M}^{\mathrm{T}}) + H(\overline{W})[\overline{K}_1\overline{W} - \overline{K}_2\nabla^2\overline{W}]\\ &\quad = \widetilde{L}(\overline{W}+\overline{W}^*,\overline{F})\end{aligned} \tag{1}$$

$$\widetilde{L}_{21}(\overline{F}) + \widetilde{L}_{22}(\overline{\Psi}_x) + \widetilde{L}_{23}(\overline{\Psi}_y) - \widetilde{L}_{24}(\overline{W}) - \widetilde{L}_{25}(\overline{N}^{\mathrm{T}}) = -\frac{1}{2}\widetilde{L}(\overline{W}+2\overline{W}^*,\overline{W}) \tag{2}$$

$$\widetilde{L}_{31}(\overline{W}) + \widetilde{L}_{32}(\overline{\Psi}_x) - \widetilde{L}_{33}(\overline{\Psi}_y) + \widetilde{L}_{34}(\overline{F}) - \widetilde{L}_{35}(\overline{N}^{\mathrm{T}}) - \widetilde{L}_{36}(\overline{S}^{\mathrm{T}}) = 0 \tag{3}$$

$$\widetilde{L}_{41}(\overline{W}) - \widetilde{L}_{42}(\overline{\Psi}_x) + \widetilde{L}_{43}(\overline{\Psi}_y) + \widetilde{L}_{44}(\overline{F}) - \widetilde{L}_{45}(\overline{N}^{\mathrm{T}}) - \widetilde{L}_{46}(\overline{S}^{\mathrm{T}}) = 0 \tag{4}$$

where $H(\overline{W})$ is the Heaviside step function and takes care of the tensionless character of the foundation

$$H(\overline{W}) = \begin{cases}1 & \overline{W} > 0\\ 0 & \overline{W} \leqslant 0\end{cases} \tag{5}$$

and the linear operators $\widetilde{L}_{ij}(\ )$ and the non-linear operator $\widetilde{L}(\ )$ are defined as in Shen (2000a).

Attention is confined to the two cases of: (1) antisymmetric angle-ply laminated plates; and (2) symmetric cross-ply laminated plates from which solutions for single-layer isotropic and orthotropic plates follow as limiting cases. Note that for these cases the plate remains flat up to the bifurcation point unless there is an initial geometric imperfection, as previously shown in Shen (1999, 2000a).

The forces, moments and higher order moments caused by elevated temperature are defined by

$$\begin{bmatrix}\overline{N}_x^{\mathrm{T}} & \overline{M}_x^{\mathrm{T}} & \overline{P}_x^{\mathrm{T}}\\ \overline{N}_y^{\mathrm{T}} & \overline{M}_y^{\mathrm{T}} & \overline{P}_y^{\mathrm{T}}\\ \overline{N}_{xy}^{\mathrm{T}} & \overline{M}_{xy}^{\mathrm{T}} & \overline{P}_{xy}^{\mathrm{T}}\end{bmatrix} = \sum_{k=1}^{N}\int_{t_{k-1}}^{t_k}\begin{bmatrix}A_x\\ A_y\\ A_{xy}\end{bmatrix}_k \Delta T(1,Z,Z^3)\,\mathrm{d}Z \tag{6a}$$

and

$$\begin{bmatrix}\overline{S}_x^{\mathrm{T}}\\ \overline{S}_y^{\mathrm{T}}\\ \overline{S}_{xy}^{\mathrm{T}}\end{bmatrix} = \begin{bmatrix}\overline{M}_x^{\mathrm{T}}\\ \overline{M}_y^{\mathrm{T}}\\ \overline{M}_{xy}^{\mathrm{T}}\end{bmatrix} - \frac{4}{3t^2}\begin{bmatrix}\overline{P}_x^{\mathrm{T}}\\ \overline{P}_y^{\mathrm{T}}\\ \overline{P}_{xy}^{\mathrm{T}}\end{bmatrix} \tag{6b}$$

in which

$$\begin{bmatrix}A_x\\ A_y\\ A_{xy}\end{bmatrix} = -\begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16}\\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26}\\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}\end{bmatrix}\begin{bmatrix}c^2 & s^2\\ s^2 & c^2\\ 2cs & -2cs\end{bmatrix}\begin{bmatrix}\alpha_{11}\\ \alpha_{22}\end{bmatrix} \tag{7}$$

where ΔT is temperature rise from a reference temperature at which there are no thermal strains, α_{11} and α_{22} are the thermal expansion coefficients in the longitudinal and transverse directions, and $\overline{Q}_{ij}$ are the transformed elastic constants, details of which can be found in Shen (1999, 2000a), and $c = \cos\theta$, $s = \sin\theta$, where θ is the lamination angle with respect to the plate X-axis.

The four edges of the plate are assumed to be simply supported and either "movable" for compressive buckling problem or "immovable" (i.e., the membrane shear force is zero and the average in-plane displacement normal to the edge is zero) for thermal buckling problem, so that the boundary conditions are

$X = 0, a$:

$$\overline{W} = \overline{\Psi}_y = 0 \tag{8a}$$

$$\overline{N}_{xy} = 0, \quad \overline{M}_x = \overline{P}_x = 0 \tag{8b}$$

$$\int_0^b \overline{N}_x \,\mathrm{d}Y + \sigma_x t b = 0 \quad \text{(for compressive buckling problem)} \tag{8c}$$

$$\int_0^b \int_0^a \frac{\partial \overline{U}}{\partial X}\,\mathrm{d}X\,\mathrm{d}Y = 0 \quad \text{(for thermal buckling problem)} \tag{8d}$$

$Y = 0, b$:

$$\overline{W} = \overline{\Psi}_x = 0 \tag{8e}$$

$$\overline{N}_{xy} = 0, \quad \overline{M}_y = \overline{P}_y = 0 \tag{8f}$$

$$\int_0^a \overline{N}_y \,\mathrm{d}X = 0 \quad \text{(for compressive buckling problem)} \tag{8g}$$

$$\int_0^a \int_0^b \frac{\partial \overline{V}}{\partial Y}\,\mathrm{d}Y\,\mathrm{d}X = 0 \quad \text{(for thermal buckling problem)} \tag{8h}$$

where σ_x is the average, externally applied axial stress, $\overline{M}_x$ and $\overline{M}_y$ are the bending moments, and $\overline{P}_x$ and $\overline{P}_y$ are the higher order moments defined as in Reddy (1984a,b).

The average end-shortening relationships are

$$\begin{aligned}\frac{\Delta_x}{a} &= -\frac{1}{ab}\int_0^b\int_0^a \frac{\partial \overline{U}}{\partial X}\,\mathrm{d}X\,\mathrm{d}Y \\ &= -\frac{1}{ab}\int_0^b\int_0^a \left\{ \left[A_{11}^* \frac{\partial^2 \overline{F}}{\partial Y^2} + A_{12}^* \frac{\partial^2 \overline{F}}{\partial X^2} + \left(B_{16}^* - \frac{4}{3t^2}E_{16}^*\right)\left(\frac{\partial \overline{\Psi}_x}{\partial Y} + \frac{\partial \overline{\Psi}_y}{\partial X}\right) - \frac{8}{3t^2}E_{16}^* \frac{\partial^2 \overline{W}}{\partial X \partial Y}\right] \right. \\ &\quad \left. - \frac{1}{2}\left(\frac{\partial \overline{W}}{\partial X}\right)^2 - \frac{\partial \overline{W}}{\partial X}\frac{\partial \overline{W}^*}{\partial X} - (A_{11}^* \overline{N}_x^{\mathrm{T}} + A_{12}^* \overline{N}_y^{\mathrm{T}}) \right\} \mathrm{d}X\,\mathrm{d}Y \end{aligned} \tag{9a}$$

$$\begin{aligned}\frac{\Delta_y}{b} &= -\frac{1}{ab}\int_0^a\int_0^b \frac{\partial \overline{V}}{\partial Y}\,\mathrm{d}Y\,\mathrm{d}X \\ &= -\frac{1}{ab}\int_0^a\int_0^b \left\{ \left[A_{22}^* \frac{\partial^2 \overline{F}}{\partial X^2} + A_{12}^* \frac{\partial^2 \overline{F}}{\partial Y^2} + \left(B_{26}^* - \frac{4}{3t^2}E_{26}^*\right)\left(\frac{\partial \overline{\Psi}_x}{\partial Y} + \frac{\partial \overline{\Psi}_y}{\partial X}\right) - \frac{8}{3t^2}E_{26}^* \frac{\partial^2 \overline{W}}{\partial X \partial Y}\right] \right. \\ &\quad \left. - \frac{1}{2}\left(\frac{\partial \overline{W}}{\partial Y}\right)^2 - \frac{\partial \overline{W}}{\partial Y}\frac{\partial \overline{W}^*}{\partial Y} - (A_{12}^* \overline{N}_x^{\mathrm{T}} + A_{22}^* \overline{N}_y^{\mathrm{T}}) \right\} \mathrm{d}Y\,\mathrm{d}X \end{aligned} \tag{9b}$$

where Δ_x and Δ_y are plate end-shortening displacements in the X- and Y-direction. In Eq. (9) and what follows, $[A^*_{ij}]$, $[B^*_{ij}]$, $[D^*_{ij}]$, $[E^*_{ij}]$, $[F^*_{ij}]$ and $[H^*_{ij}]$ $(i, j = 1, 2, 6)$ are the reduced stiffness matrices and determined through relationships (Shen, 1999, 2000a)

$$\mathbf{A}^* = \mathbf{A}^{-1}, \quad \mathbf{B}^* = -\mathbf{A}^{-1}\mathbf{B}, \quad \mathbf{D}^* = \mathbf{D} - \mathbf{B}\mathbf{A}^{-1}\mathbf{B}, \quad \mathbf{E}^* = -\mathbf{A}^{-1}\mathbf{E}, \quad \mathbf{F}^* = \mathbf{F} - \mathbf{E}\mathbf{A}^{-1}\mathbf{B},$$
$$\mathbf{H}^* = \mathbf{H} - \mathbf{E}\mathbf{A}^{-1}\mathbf{E} \tag{10}$$

where A_{ij}, B_{ij} etc., are the plate stiffnesses, defined in the standard way.

3. Analytical method and solution procedure

Having developed the theory, one is in a position to solve Eqs. (1)–(4) with boundary conditions (8). Before proceeding, it is convenient to first define the following dimensionless quantities for such plates (with γ_{ijk} defined as in Shen (1999)), in which the alternative forms k_1 and k_2 are not needed until the numerical examples are considered.

$$\begin{aligned}
&x = \pi X/a, \quad y = \pi Y/b, \quad \beta = a/b, \quad (W, W^*) = (\overline{W}, \overline{W}^*)/[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/4} \\
&F = \overline{F}/[D^*_{11}D^*_{22}]^{1/2}, \quad (\Psi_x, \Psi_y) = (\overline{\Psi}_x, \overline{\Psi}_y)a/\pi[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/4} \\
&\gamma_{14} = [D^*_{22}/D^*_{11}]^{1/2}, \quad \gamma_{24} = [A^*_{11}/A^*_{22}]^{1/2}, \quad \gamma_5 = -A^*_{12}/A^*_{22} \\
&(\gamma_{T1}, \gamma_{T2}) = (A^{\mathrm{T}}_x, A^{\mathrm{T}}_y)a^2/\alpha_0\pi^2[D^*_{11}D^*_{22}]^{1/2} \\
&(M_x, M_y, P_x, P_y) = (\overline{M}_x, \overline{M}_y, 4\overline{P}_x/3t^2, 4\overline{P}_y/3t^2)a^2/\pi^2 D^*_{11}[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/4} \\
&(K_1, k_1) = \overline{K}_1(a^4/\pi^4 D^*_{11}, b^4/E_{22}t^3), \quad (K_2, k_2) = \overline{K}_2(a^2/\pi^2 D^*_{11}, b^2/E_{22}t^3) \\
&(\delta_x, \delta_y) = (\Delta_x/a, \Delta_y/b)b^2/4\pi^2[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/2} \\
&\lambda_x = \sigma_x b^2 t/4\pi^2[D^*_{11}D^*_{22}]^{1/2}, \quad \lambda_T = \alpha_0 \Delta T
\end{aligned} \tag{11}$$

Also let the thermal expansion coefficients for each ply be

$$\alpha_{11} = a_{11}\alpha_0, \quad \alpha_{22} = a_{22}\alpha_0 \tag{12}$$

where α_0 is an arbitrary reference value, and let

$$(A^{\mathrm{T}}_x, A^{\mathrm{T}}_y) = -\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (A_x, A_y)_k \, \mathrm{d}Z \tag{13}$$

Eqs. (1)–(4) may then be written in dimensionless form as

$$L_{11}(W) - L_{12}(\Psi_x) - L_{13}(\Psi_y) + \gamma_{14}L_{14}(F) + H(W)[K_1 W - K_2\nabla^2 W] = \gamma_{14}\beta^2 L(W + W^*, F) \tag{14}$$

$$L_{21}(F) + \gamma_{24}L_{22}(\Psi_x) + \gamma_{24}L_{23}(\Psi_y) - \gamma_{24}L_{24}(W) = -\frac{1}{2}\gamma_{24}\beta^2 L(W + 2W^*, W) \tag{15}$$

$$L_{31}(W) + L_{32}(\Psi_x) - L_{33}(\Psi_y) + \gamma_{14}L_{34}(F) = 0 \tag{16}$$

$$L_{41}(W) - L_{42}(\Psi_x) + L_{43}(\Psi_y) + \gamma_{14}L_{44}(F) = 0 \tag{17}$$

where the non-dimensional linear operators $L_{ij}(\ )$ and the non-linear operator $L(\ )$ are defined as in Shen (1999).

The boundary conditions of Eq. (8) become

$x = 0, \pi$:

$$W = \Psi_y = 0 \tag{18a}$$

$$F_{,xy} = 0, \quad M_x = P_x = 0 \tag{18b}$$

$$\frac{1}{\pi}\int_0^\pi \beta^2 \frac{\partial^2 F}{\partial y^2}\,\mathrm{d}y + 4\lambda_x\beta^2 = 0 \quad \text{(for compressive buckling problem)} \tag{18c}$$

$$\delta_x = 0 \quad \text{(for thermal buckling problem)} \tag{18d}$$

$y = 0, \pi$:

$$W = \Psi_x = 0 \tag{18e}$$

$$F_{,xy} = 0, \quad M_y = P_y = 0 \tag{18f}$$

$$\int_0^\pi \frac{\partial^2 F}{\partial x^2}\,\mathrm{d}x = 0 \quad \text{(for compressive buckling problem)} \tag{18g}$$

$$\delta_y = 0 \quad \text{(for thermal buckling problem)} \tag{18h}$$

and the unit end-shortening relationships become

$$\begin{aligned}\delta_x = -\frac{1}{4\pi^2\beta^2\gamma_{24}}\int_0^\pi\int_0^\pi \Bigg\{ &\left[\gamma_{24}^2\beta^2\frac{\partial^2 F}{\partial y^2} - \gamma_5\frac{\partial^2 F}{\partial x^2} + \gamma_{24}\gamma_{223}\left(\beta\frac{\partial \Psi_x}{\partial y} + \frac{\partial \Psi_y}{\partial x}\right) - 2\gamma_{24}\gamma_{516}\beta\frac{\partial^2 W}{\partial x\partial y}\right] \\ &-\frac{1}{2}\gamma_{24}\left(\frac{\partial W}{\partial x}\right)^2 - \gamma_{24}\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x} + (\gamma_{24}^2\gamma_{T1} - \gamma_5\gamma_{T2})\lambda_T\Bigg\}\mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{19a}$$

$$\begin{aligned}\delta_y = -\frac{1}{4\pi^2\beta^2\gamma_{24}}\int_0^\pi\int_0^\pi \Bigg\{ &\left[\frac{\partial^2 F}{\partial x^2} - \gamma_5\beta^2\frac{\partial^2 F}{\partial y^2} + \gamma_{24}\gamma_{230}\left(\beta\frac{\partial \Psi_x}{\partial y} + \frac{\partial \Psi_y}{\partial x}\right) - 2\gamma_{24}\gamma_{526}\beta\frac{\partial^2 W}{\partial x\partial y}\right] \\ &-\frac{1}{2}\gamma_{24}\beta^2\left(\frac{\partial W}{\partial y}\right)^2 - \gamma_{24}\beta^2\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y} + (\gamma_{T2} - \gamma_5\gamma_{T1})\lambda_T\Bigg\}\mathrm{d}y\,\mathrm{d}x\end{aligned} \tag{19b}$$

With the use of Eqs. (14)–(19), the postbuckling response of shear deformable laminated plates resting on a tensionless elastic foundation subjected to uniaxial compression or a uniform temperature rise is now determined by means of a two step perturbation technique, for which the small perturbation parameter has no physical meaning at the first step, and is then replaced by a dimensionless deflection at the second step. The essence of this procedure, in the present case, is to assume that

$$\begin{aligned}W(x,y,\varepsilon) &= \sum_{j=1}\varepsilon^j w_j(x,y), \quad F(x,y,\varepsilon) = \sum_{j=0}\varepsilon^j f_j(x,y) \\ \Psi_x(x,y,\varepsilon) &= \sum_{j=1}\varepsilon^j \psi_{xj}(x,y), \quad \Psi_y(x,y,\varepsilon) = \sum_{j=1}\varepsilon^j \psi_{yj}(x,y)\end{aligned} \tag{20}$$

where ε is a small perturbation parameter and the first term of $w_j(x,y)$ is assumed to have the form

$$w_1(x,y) = A_{11}^{(1)}\sin mx\,\sin ny \tag{21}$$

The initial geometric imperfection is assumed to have the similar form

$$W^*(x,y,\varepsilon) = \varepsilon a_{11}^* \sin mx \sin ny = \varepsilon\mu A_{11}^{(1)} \sin mx \sin ny \tag{22}$$

where $\mu = a_{11}^*/A_{11}^{(1)}$ is the imperfection parameter.

All the necessary steps of the solution methodology are described below, but the solutions are not repeated here for convenience, with the minor changes included that are needed to make them applicable to tensionless foundations instead of conventional foundations.

First, the assumed solution form of Eq. (20) is substituted into Eqs. (14)–(17) to obtain a set of perturbation equations by collecting the terms of the same order of ε.

Then, Eqs. (21) and (22) are used to solve these perturbation equations of each order step by step. At each step the amplitudes of the terms $w_j(x,y)$, $f_j(x,y)$, $\psi_{xj}(x,y)$, and $\psi_{yj}(x,y)$ can be determined, e.g. $B_{11}^{(1)}$, $B_{20}^{(2)}$, $B_{02}^{(2)}$, etc., except for $B_{00}^{(j)}$ $(j = 0, 2, 4, \dots)$ and which can be determined by the Galerkin procedure. As a result, the large deflection asymptotic solutions of $W(x,y)$, $F(x,y)$, $\Psi_x(x,y)$ and $\Psi_y(x,y)$ are obtained.

Next, upon substitution of $W(x,y)$, $F(x,y)$, $\Psi_x(x,y)$ and $\Psi_y(x,y)$ into Eqs. (18c) and (19a) or Eqs. (18d) and (18h), the postbuckling equilibrium paths for the plate subjected to uniaxial compression can be written as

$$\lambda_x = \lambda_x^{(0)} + \lambda_x^{(2)} W_m^2 + \lambda_x^{(4)} W_m^4 + \cdots \tag{23a}$$

$$\delta_x = \delta_x^{(0)} + \delta_x^{(2)} W_m^2 + \delta_x^{(4)} W_m^4 + \cdots \tag{23b}$$

and for the plate subjected to a uniform temperature rise can be written as

$$\lambda_{\mathrm{T}} = \lambda_{\mathrm{T}}^{(0)} + \lambda_{\mathrm{T}}^{(2)} W_m^2 + \lambda_{\mathrm{T}}^{(4)} W_m^4 + \cdots \tag{24}$$

in which W_m is the dimensionless form of maximum deflection, and $\lambda_x^{(i)}$, $\delta_x^{(i)}$ and $\lambda_{\mathrm{T}}^{(i)}$ $(i = 0, 2, 4, \dots)$ are given in detail in Appendix A.

Eqs. (23) and (24) can be employed to obtain numerical results for full non-linear postbuckling load-end shortening and/or load-deflection curves of shear deformable laminated plates subjected to uniaxial compression or a uniform temperature rise and resting on a tensionless elastic foundation. Since the foundation reacts in compression only, a possible uplifting region is expected. The solution procedure is complicated and therefore an iterative procedure is necessary to solve this strong non-linear problem. In applying the contact condition, the plate area is discretized into a series of grids, and the contact status is assessed at each grid location. From Eq. (A.2) in Appendix A one can see some equations, e.g. Q_{11}, Q_{13} and Q_{31} involving K_1, K_2 and the contact function $H[W(x_g,y_g)]$, where $W(x_g,y_g)$ is the deflection at the grid coordinate (x_g,y_g) and summation is carried out over all grid coordinates by using the Gauss-Legendre quadrature procedure with Gauss weight assigned $C_g^{(M)}$. It is found that an acceptable accuracy can be obtained by taking into account 20×20 points, which is employed in the next section.

As is well known, the instability phenomenon is designed as bifurcation, since it occurs with the bifurcation of a new equilibrium path from the original one. The buckling load (or buckling temperature) of perfect plates can be obtained by setting $\mu = 0$ (or $\overline{W}^*/t = 0$), while taking $W_m = 0$ (or $\overline{W}/t = 0$). Based on Eq. (5), when $W_m = 0$ the buckling load for unilaterally constrained plate is identical to that of the unconstrained plate. Hence we define λ_x $(W_m = 0^+)$ as the buckling load for the plate resting on tensionless elastic foundations. The minimum buckling load (or buckling temperature) is determined by considering Eq. (23a) or (24) for various combinations of m and n which determine the numbers of half-waves in the X- and Y-directions respectively.

4. Numerical examples and discussions

Numerical results are presented in this section for perfect and imperfect, unilaterally constrained shear deformable laminated plates where the outmost layer is the first mentioned orientation. A computer program was developed for this purpose and many examples have been solved numerically, including the following.

The accuracy and effectiveness of the present method for the compressive postbuckling and thermal postbuckling behaviors of shear deformable laminated plates with or without elastic foundations were examined by many comparison studies as previously given in Shen (1999, 2000a,b,c, 2001). In addition, the postbuckling response for an isotropic thin square plate subjected to uniaxial compression are calculated and compared in Table 1 with the analytical solutions of Dym (1974) and FEM results of Sundaresan et al. (1996). Then the thermal postbuckling response for a $(\pm 45)_{2T}$ laminated rectangular plate subjected to a uniform temperature rise are calculated and compared in Table 2 with the FEM results of Thankam et al. (2003). The material properties adopted here are: $E_{11}/E_{22} = 25$, $G_{12}/E_{22} = G_{13}/E_{22} = 0.5$, $G_{23}/E_{22} = 0.2$, $\nu_{12} = 0.25$ and $\alpha_{22}/\alpha_{11} = 10$. These comparisons show that the results from the present method are in good agreement with the existing results, thus verifying the reliability and accuracy of the present method.

A parametric study intended to supply information on the postbuckling response of unilaterally constrained shear deformable laminated plates subjected to uniaxial compression or a uniform temperature rise was undertaken. For all of the examples the plate width-to-thickness ratio $b/t = 20$, all plies are of equal thickness and the material properties adopted as in Lee and Lee (1997) are: $E_{11} = 155$ GPa, $E_{22} = 8.07$ GPa, $G_{12} = G_{13} = 4.55$ GPa, $G_{23} = 3.25$ GPa, $\nu_{12} = 0.22$, $\alpha_{11} = -0.07 \times 10^{-6}/°C$ and $\alpha_{22} = 30.1 \times 10^{-6}/°C$. Typical results are shown in Figs 1–12, in which $\bar{\lambda}_x = \sigma_x (b/t)^2/E_{22}$ and $\bar{\lambda}_T = \alpha_{22}\Delta T (b/t)^2$. It is noted that in Figs. 1–8, for the plate resting on tensionless elastic foundations the buckling load is obtained by using $\overline{W}/t = 1.0\mathrm{e}-4$. In all these figures $\overline{W}^*/t$ and $\overline{W}/t$ mean the dimensionless forms of, respectively, the maximum initial geometric imperfection and additional deflections of the plate.

Figs. 1 and 2 give, respectively, the postbuckling load-deflection and load-shortening curves of $(\pm 45)_{2T}$ laminated rectangular plates with $\beta = 5$ resting on tensionless and conventional elastic foundations of the Pasternak-type subjected to uniaxial compression. Two different values of foundation stiffnesses $(k_1, k_2) = (10, 1)$ and $(100, 10)$ are considered. The results for the same unconstrained plate (referred to as "foundationless plates" in the figures) are also given as comparators. The buckling loads of the plate resting on tensionless elastic foundations with $(k_1, k_2) = (10, 1)$ and $(100, 10)$, compared to the buckling load of the unconstrained plate, represent increases of about 4.8% and 34%, respectively. This increase becomes greater as the foundation stiffness is increased, or vice versa. The results also show that the postbuckling load-deflection curve for the plate resting on a tensionless elastic foundation lies between the two of the unconstrained plate and the plate resting on a conventional elastic foundation.

Table 1
Comparisons of postbuckling response for an isotropic thin square plate subjected to uniaxial compression ($\nu = 0.3$)

$\overline{W}/t$	$\lambda_x/(\lambda_x)_{cr}$: Sundaresan et al. (1996)	Dym (1974)	Present
0.0	1.0	1.0	1.0
0.2	1.0137	1.0137	1.0137
0.4	1.0547	1.0546	1.0547
0.6	1.1233	1.1229	1.1232
0.8	1.2198	1.2184	1.2196
1.0	1.3445	1.3413	1.3443
$\sigma_{cr} b^2 t/\pi^2 D$	4.0081	4.0	4.0

Table 2
Comparisons of thermal postbuckling response for a $(\pm 45)_{2T}$ laminated rectangular plate subjected to a uniform temperature rise ($b/t = 100$, $E_{11}/E_{22} = 25$, $G_{12}/E_{22} = G_{13}/E_{22} = 0.5$, $G_{23}/E_{22} = 0.2$, $\nu_{12} = 0.25$ and $\alpha_{22}/\alpha_{11} = 10$)

$\lambda_T/(\lambda_T)_{cr}$ $\overline{W}/t$	$\beta = 0.75$		$\beta = 1.0$		$\beta = 1.5$	
	Thankam et al. (2003)	Present	Thankam et al. (2003)	Present	Thankam et al. (2003)	Present
0.0	1.0	1.0	1.0	1.0	1.0	1.0
0.2	1.042	1.0412	1.039	1.0392	1.043	1.0427
0.4	1.167	1.1649	1.158	1.1570	1.174	1.1713
0.6	1.376	1.3720	1.356	1.3539	1.393	1.3868
0.8	1.670	1.6640	1.635	1.6309	1.702	1.6911
1.0	2.053	2.0426	1.995	1.9892	2.101	2.0866
$\alpha_{22}(\Delta T)_{cr}(b/t)^2$	12.709	12.649	9.493	9.4584	6.410	6.3939

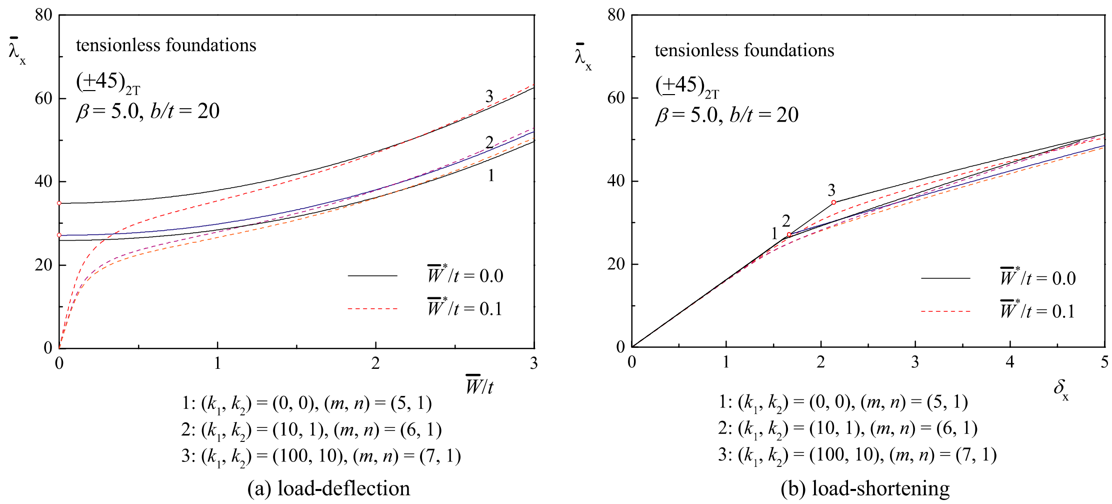
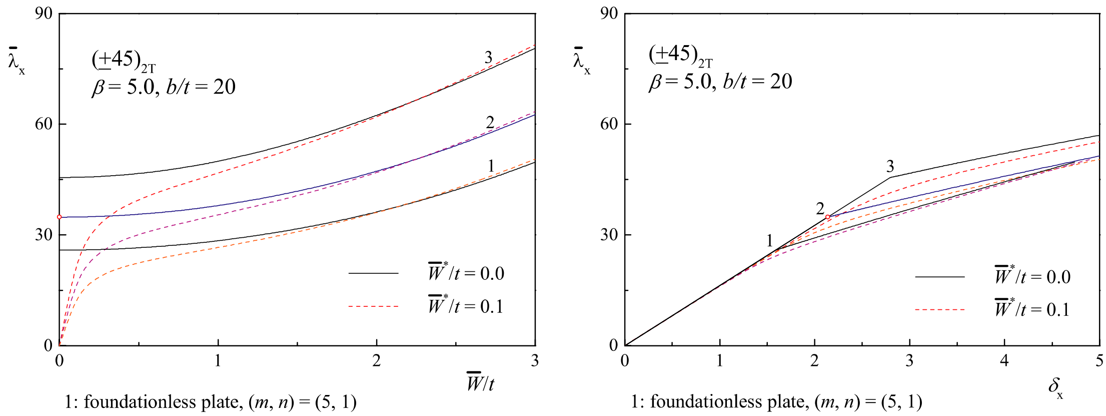


Fig. 1. Postbuckling behavior for a $(\pm 45)_{2T}$ rectangular plate resting on tensionless foundations.

Figs. 3 and 4 give, respectively, the postbuckling load-deflection and load-shortening curves of $(0/90)_S$ laminated rectangular plates with $\beta = 5$ resting on tensionless and conventional elastic foundations of the Pasternak-type subjected to uniaxial compression. It can be seen that the foundation stiffness affects the postbuckling response of the $(0/90)_S$ plate more than that of the $(\pm 45)_{2T}$ one. In the present example the buckling loads of the plate resting on tensionless elastic foundations with $(k_1, k_2) = (10, 1)$ and $(100, 10)$, compared to the buckling load of the unconstrained plate, represent increases of about 13.9% and 111%, respectively. It can also be seen that the load-shortening curve of the plate resting on a tensionless elastic foundation with $(k_1, k_2) = (100, 10)$ becomes lower than that of the unconstrained plate when $\delta_x > 2$.

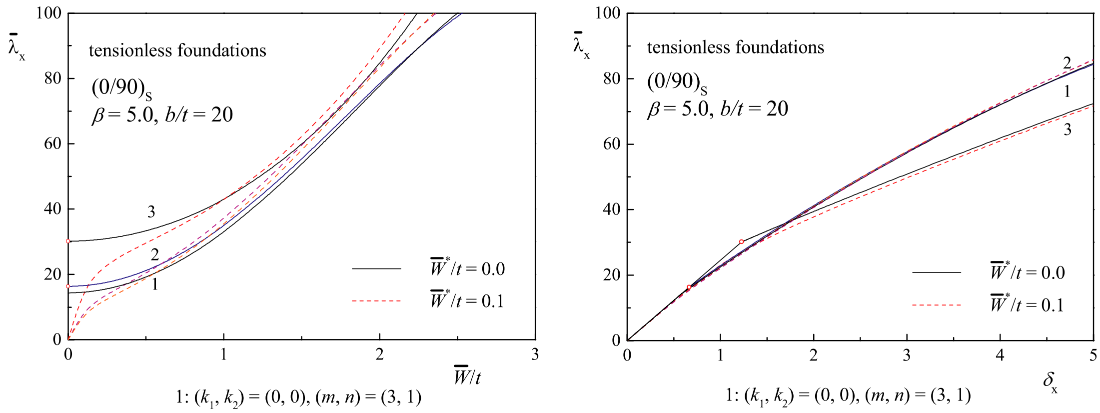
It is noted that we choose $\beta = 5$ here because for the low value of the plate aspect ratio, e.g. $\beta = 1$, there are no positive deflections and no contact region is expected under the mechanical loading conditions. In



1: foundationless plate, $(m, n) = (5, 1)$
2: tensionless foundation $(k_1, k_2) = (100, 10)$, $(m, n) = (7, 1)$
3: conventional foundation $(k_1, k_2) = (100, 10)$, $(m, n) = (9, 1)$

(a) load-deflection (b) load-shortening

Fig. 2. Postbuckling behavior for a $(\pm 45)_{2T}$ rectangular plate resting on conventional and tensionless foundations.



1: $(k_1, k_2) = (0, 0)$, $(m, n) = (3, 1)$
2: $(k_1, k_2) = (10, 1)$, $(m, n) = (3, 1)$
3: $(k_1, k_2) = (100, 10)$, $(m, n) = (5, 1)$

(a) load-deflection (b) load-shortening

Fig. 3. Postbuckling behavior of a $(0/90)_S$ rectangular plate resting on tensionless foundations.

contrast, even for the square plate subjected to thermal loading the positive deflection may occur and a possible contact region is expected as shown in the next example.

Figs. 5 and 6 give, respectively, the thermal postbuckling load-deflection curves of $(\pm 15)_{2T}$ laminated square plates resting on tensionless and conventional elastic foundations of the Pasternak-type subjected to a uniform temperature rise. Now the plate buckles with $(m, n) = (1, 2)$ and a possible contact region is expected. The thermal buckling loads of the plate resting on tensionless elastic foundations with

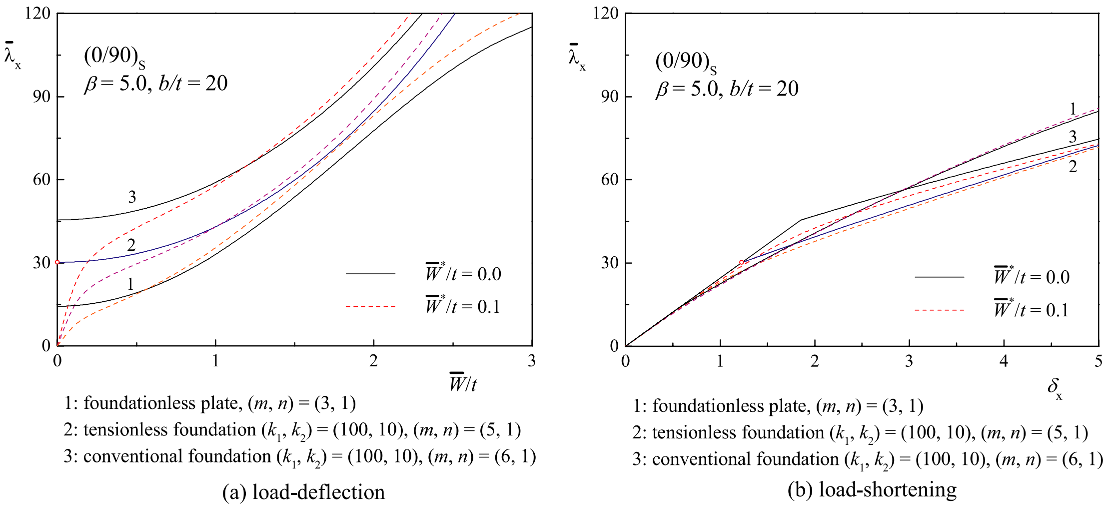


Fig. 4. Postbuckling behavior of a $(0/90)_S$ rectangular plate resting on conventional and tensionless foundations.

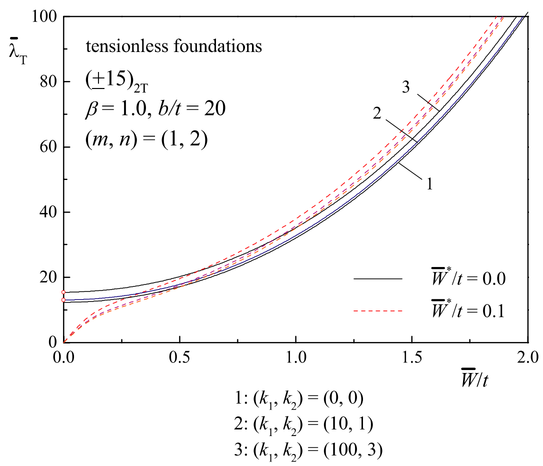


Fig. 5. Thermal postbuckling load-deflection curves for a $(\pm 15)_{2T}$ square plate resting on tensionless foundations.

$(k_1, k_2) = (10, 1)$ and $(100, 3)$, compared to the thermal buckling load of the unconstrained plate, represent increases of about 6% and 25%, respectively.

Figs. 7 and 8 give, respectively, the thermal postbuckling load-deflection curves of $(90/0)_S$ laminated rectangular plates with $\beta = 5$ resting on tensionless and conventional elastic foundations of the Pasternak-type subjected to a uniform temperature rise. In this case, the unconstrained plate and the plate resting on a tensionless elastic foundation with $(k_1, k_2) = (10, 1)$ have buckling mode $(m, n) = (6, 1)$, and for the plate resting on tensionless and conventional elastic foundations with $(k_1, k_2) = (100, 3)$ the plate buckles with $(m, n) = (7, 1)$. Note that for these two plates, $(\pm 15)_{2T}$ and $(90/0)_S$, when $(k_1, k_2) = (100, 10)$ the plates will have buckling mode $(m, n) = (1, 1)$ and no contact region is expected.

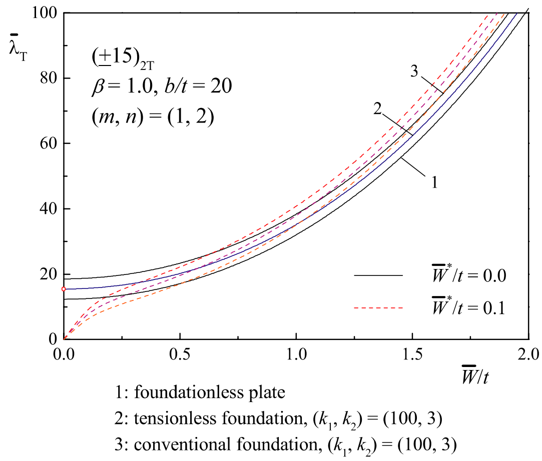


Fig. 6. Thermal postbuckling load-deflection curves for a $(\pm 15)_{2T}$ square plate resting on conventional and tensionless foundations.

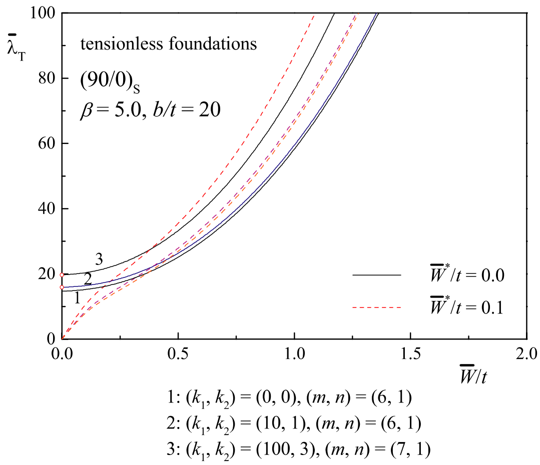


Fig. 7. Thermal postbuckling load-deflection curves for a $(90/0)_S$ rectangular plate resting on tensionless foundations.

Postbuckling load-shortening and/or load-deflection curves for imperfect as well as perfect plates are plotted in each of Figs. 1–8. The imperfect curves show that the effect of an initial geometric imperfection on the postbuckling response is substantial. This conclusion is valid for the plate resting on both tensionless and conventional elastic foundations.

Fig. 9 shows the deformed shapes of perfect $(\pm 45)_{2T}$ and $(0/90)_S$ plates resting on tensionless elastic foundations of three different values of k_1 and k_2 in the postbuckling range ($\overline{W}/t = 1.0$). It can be seen that the contact area increases slightly as foundation stiffness increases. The transverse displacements in the contact regions are smaller than those in the non-contact regions. If the foundation is quite rigid, e.g. $(k_1, k_2) = (100, 10)$ and $(200, 20)$ in this example, there are no transverse displacements in the contact regions. Fig. 10 shows the deformed shapes of unilaterally constrained perfect $(\pm 45)_{2T}$ and $(0/90)_S$ plates at different points of the postbuckling path ($\overline{W}/t = 0.5$, 0.75, 1.0). It can be seen that the difference between

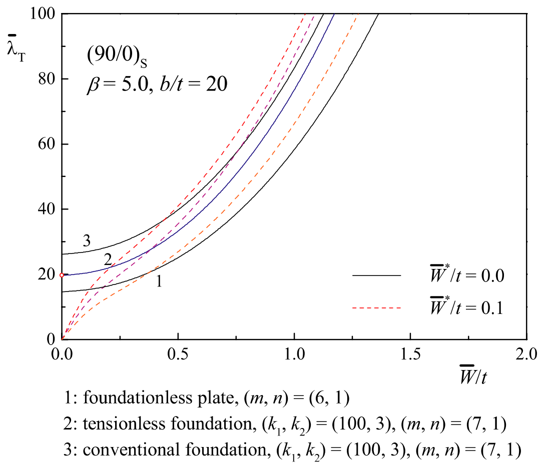


Fig. 8. Thermal postbuckling load-deflection curves for a $(90/0)_S$ rectangular plate resting on conventional and tensionless foundations.

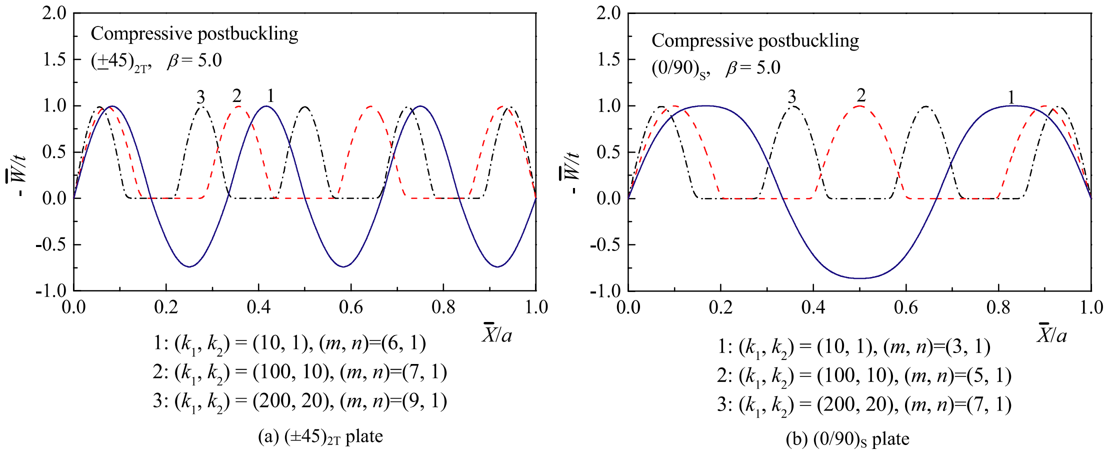


Fig. 9. Deformed shapes of perfect laminated plates with different values of foundation stiffness in the postbuckling range.

the displacements in the contact and non-contact regions decreases as the applied load is increased. Note that in Fig. 10, the mode of postbuckling deformation is unchanged, i.e., $(m, n) = (6, 1)$ for the $(\pm 45)_{2T}$ plate and $(m, n) = (3, 1)$ for the $(0/90)_S$ plate. The results also show that the contact area remains constant when the deflection increases from $\overline{W}/t = 0.5$ to 1.0. In reality, mode changes are possible in the deep postbuckling range (Chai, 2001), and as a result the contact region expands as the plate deflection is increased and the effect of unilateral constraint becomes more pronounced.

Figs. 11 and 12 are thermal postbuckling results of $(90/0)_S$ plates analogous to the compressive postbuckling results of Figs. 9 and 10. They lead to broadly the same conclusions as do Figs. 9 and 10.

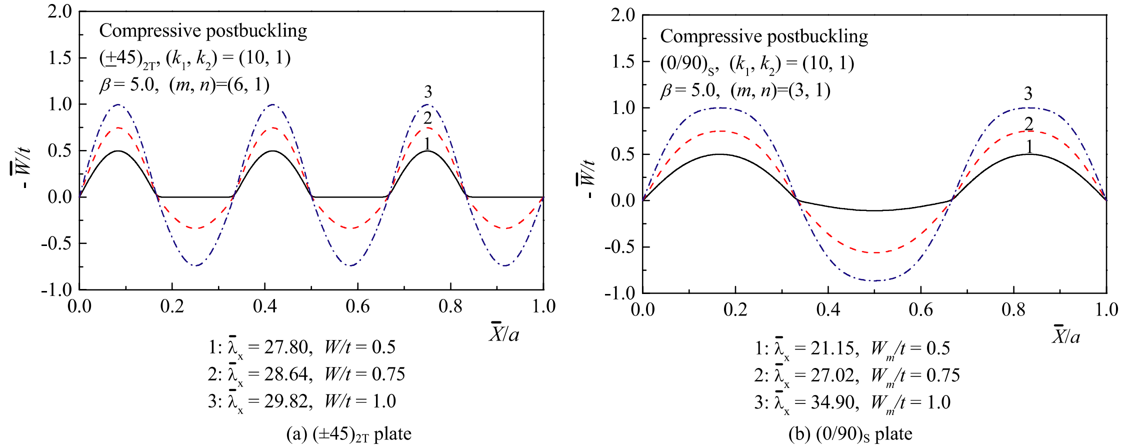


Fig. 10. Deformed shapes of perfect laminated plates supported by a tensionless foundation in the postbuckling range.

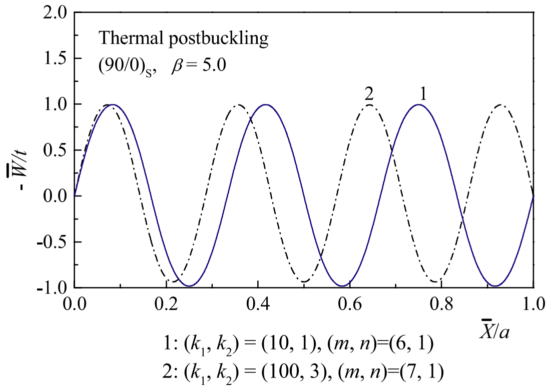


Fig. 11. Deformed shapes of perfect $(90/0)_S$ plates with different values of foundation stiffness in the thermal postbuckling range.

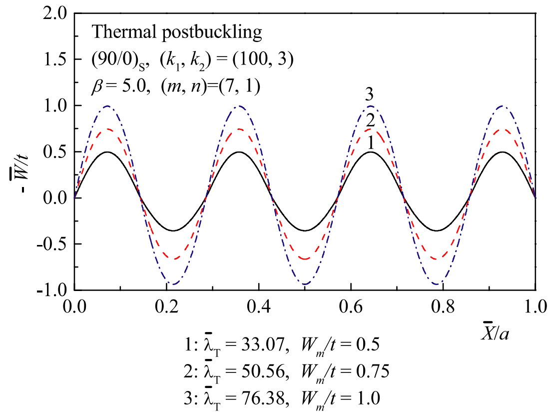


Fig. 12. Deformed shapes of perfect $(90/0)_S$ plates supported by a tensionless foundation in the thermal postbuckling range.

5. Concluding remarks

Postbuckling analysis of shear deformable laminated plates supported by a tensionless elastic foundation of the Pasternak-type subjected to in-plane compressive edge loads or a uniform temperature rise has been presented by using an analytical–numerical method. The advantage of the present method is that the solution is in an explicit form which is easy to program in computing full non-linear load-end shortening and/or load-deflection curves without any prior assumption for the shape of the contact region. A parametric study for perfect and imperfect, antisymmetric angle-ply and symmetric cross-ply laminated plates has been carried out. The numerical results showed that the unilateral constraint has a significant effect on the postbuckling response of the plate subjected to either mechanical or thermal loading when the foundation stiffness is sufficiently large.

Acknowledgement

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Appendix A

In Eqs. (23) and (24)

$$
\begin{aligned}
&(\lambda_x^{(0)}, \lambda_x^{(2)}, \lambda_x^{(4)}) = \frac{1}{4\beta^2\gamma_{14}C_{11}}(S_0, S_2, S_4), \quad (\lambda_T^{(0)}, \lambda_T^{(2)}, \lambda_T^{(4)}) = \frac{1}{\gamma_{14}C_{11}}(S_0, S_2, S_4) \\
&\delta_x^{(0)} = \gamma_{24}\lambda_x, \quad \delta_x^{(2)} = \frac{1}{32\beta^2}C_{11}(1+2\mu) \\
&\delta_x^{(4)} = \frac{1}{256\beta^2}\gamma_{14}\gamma_{24}C_{11}^2\left(\frac{m^4}{\gamma_7 J_{13}} + \frac{n^4\beta^4}{\gamma_6 J_{31}}\right)(1+\mu)^2(1+2\mu)^2
\end{aligned}
\tag{A.1}
$$

in which (with g_{ij} and g_{ijk} defined as in Shen (1999))

$$
\begin{aligned}
&S_0 = \frac{Q_{11}}{(1+\mu)}, \quad S_2 = \frac{1}{16}\gamma_{14}\gamma_{24}\Theta_2(1+2\mu), \quad S_4 = \frac{1}{256}\gamma_{14}^2\gamma_{24}^2 C_{11}(C_{24} - C_{44}) \\
&C_{24} = 2(1+\mu)^2(1+2\mu)^2\Theta_2\left(\frac{m^4}{\gamma_7 J_{13}} + \frac{n^4\beta^4}{\gamma_6 J_{31}}\right) \\
&C_{44} = (1+\mu)(1+2\mu)[2(1+\mu)^2 + (1+2\mu)]\left(\frac{m^8}{\gamma_7^2 J_{13}} + \frac{n^8\beta^8}{\gamma_6^2 J_{31}}\right) \\
&J_{13} = Q_{13}C_{11}(1+\mu) - Q_{11}C_{13}, \quad J_{31} = Q_{31}C_{11}(1+\mu) - Q_{11}C_{31} \\
&Q_{11} = \Theta_{11} + \sum_{g=0}^{M} C_g^{(M)} H[W(x_g, y_g)][K_1 + K_2(m^2 + n^2\beta^2)] \\
&Q_{13} = \Theta_{13} + \sum_{g=0}^{M} C_g^{(M)} H[W(x_g, y_g)][K_1 + K_2(m^2 + 9n^2\beta^2)] \\
&Q_{31} = \Theta_{31} + \sum_{g=0}^{M} C_g^{(M)} H[W(x_g, y_g)][K_1 + K_2(9m^2 + n^2\beta^2)]
\end{aligned}
$$

$$\Theta_{11} = g_{08} + \gamma_{14}\gamma_{24}m^2n^2\beta^2\frac{g_{05}g_{07}}{g_{06}}, \quad \Theta_2 = \frac{m^4}{\gamma_7} + \frac{n^4\beta^4}{\gamma_6} + C_{22}$$
$$\Theta_{13} = g_{138} + \gamma_{14}\gamma_{24}9m^2n^2\beta^2\frac{g_{135}g_{137}}{g_{136}}, \quad \Theta_{31} = g_{318} + \gamma_{14}\gamma_{24}9m^2n^2\beta^2\frac{g_{315}g_{317}}{g_{316}} \tag{A.2}$$
$$\gamma_6 = 1 + \gamma_{14}\gamma_{24}\gamma_{230}^2\frac{4m^2}{\gamma_{41} + \gamma_{322}4m^2}, \quad \gamma_7 = \gamma_{24}^2 + \gamma_{14}\gamma_{24}\gamma_{223}^2\frac{4n^2\beta^2}{\gamma_{31} + \gamma_{322}4n^2\beta^2}$$

in the above equations, for the case of uniaxial compression

$$C_{11} = C_{13} = m^2, \quad C_{31} = 9m^2, \quad C_{22} = 0 \tag{A.3}$$

and for the uniform temperature loading case

$$C_{11} = (\gamma_{T1}m^2 + \gamma_{T2}n^2\beta^2), \quad C_{13} = (\gamma_{T1}m^2 + 9\gamma_{T2}n^2\beta^2), \quad C_{31} = (9\gamma_{T1}m^2 + \gamma_{T2}n^2\beta^2)$$
$$C_{22} = 2\frac{(m^4 + \gamma_{24}^2n^4\beta^4) + 2\gamma_5m^2n^2\beta^2}{\gamma_{24}^2 - \gamma_5^2} \tag{A.4}$$

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